(Keep your homework - Quiz today!)

Notation: $\partial_y f = \frac{\partial f}{\partial y} = f_y$ etc.

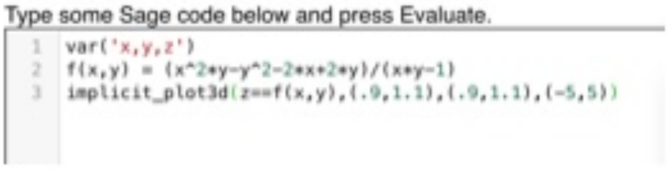
2.14b $\lim_{(x,y)\to(0,0)} \frac{\sin(3y)\sin(xy)}{xy^2} = \lim_{(x,y)\to(0,0)} \underbrace{\frac{3\sin(3y)}{3y}}_{\to 1} \cdot \underbrace{\frac{\sin(\overset{\to 0}{xy})}{\underset{\to 0}{xy}}}_{1}$

$= \boxed{3}$

Special limit $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

2.14c $\lim_{(x,y)\to(1,1)} \frac{x^2y - y^2 - 2x + 2y}{xy - 1}$ ($\frac{0}{0}$ form.)

↖ doesn't reduce



Type some Sage code below and press Evaluate.

```
var('x,y,z')
f(x,y) = (x^2*y-y^2-2*x+2*y)/(x*y-1)
implicit_plot3d(z==f(x,y),(.9,1.1),(.9,1.1),(-5,5))
```

← to see plot near $(x,y) = (1,1)$.

To show the limit does not exist, let's take the limit as we approach $(1,1)$ along different curves.

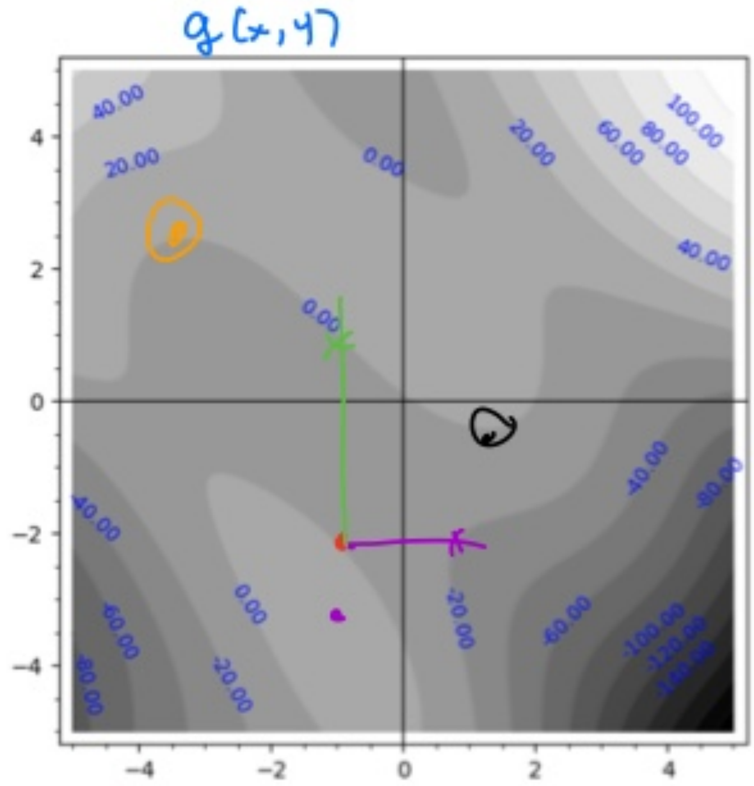
(#1) along $y = 1$

$$\lim_{(x,1)\to(1,1)} \frac{\overset{x^2-2x+1}{x^2 - 1 - 2x + 2}}{x - 1} = \lim_{x\to 1} \frac{(x-1)^2}{x-1} = \lim_{x\to 1}(x-1) = \boxed{0}.$$

(#2) along $y = x$.

$$\lim_{(x,x)\to(1,1)} \frac{x^3 - x^2 - \cancel{2x} + \cancel{2x}}{x^2 - 1}$$

$$= \lim_{x\to 1} \frac{x^2\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \boxed{\frac{1}{2}} \neq 0.$$ limit does not exist!



Estimate:

① $\partial_x g, \partial_y g$ at $(-1,-2)$

② Find 3 examples of points where $\frac{\partial g}{\partial x}=0$

③ Find a point where the tangent plane to $z=g(x,y)$ is parallel to the xy plane

④ Find a point where $\partial_y g$ is maximum.

① $\partial_x g \simeq \frac{\Delta g}{\Delta x} = \frac{-20-0}{1.6} \approx \frac{-20}{1.6} \simeq -12.5 \approx \boxed{-10}$

$\partial_y g \simeq \frac{\Delta g}{\Delta y}$ = negative (can't estimate well from given info).

② $\frac{\partial g}{\partial x}=0 \Leftrightarrow g$ is roughly constant as we move in x-direction.

$(-1,-3)$ probably close to peak: both $\partial_x g$ & $\partial_y g = 0$ there.

$(1.3,-.5)$ Contour line is parallel to x direction at the that pt.
$\Rightarrow \partial_x g=0$ there.

$(-3.5,2.5)$ also

many more.

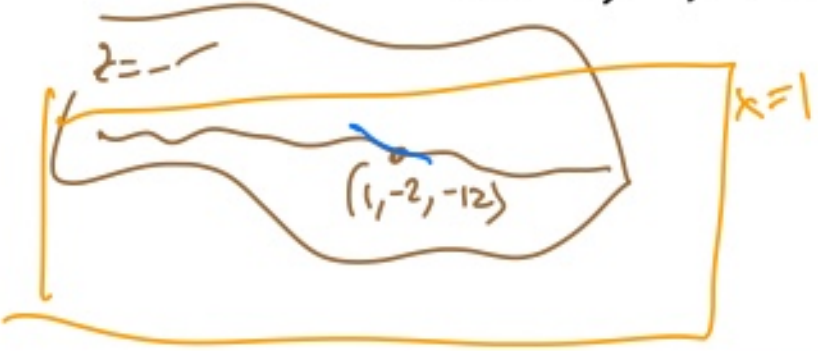
③ Point tangent plane to $z=g(x,y)$ is $\parallel$ to xy plane.

Looking for

$(-1,-3)$ would be an example.

④ $\partial_y g$ is max where contours are close in y-direction & increase as we go in +y direction.
$(x,y) \approx (5,-5)$.

Example Find the slope of the curve that is the intersection of $z = 4x^2y - y^2x^3$ and $x=1$ at $(1,-2,-12)$.



$$Z_y = 4x^2 - 2yx^3 = 4 - 2(-2)(1)^3 = \boxed{8}.$$

Tangent Plane at (x_0, y_0, z_0) of $z = f(x,y)$

$$z = \underset{\substack{\uparrow \\ Z_x \\ \uparrow \\ \text{slope in x-direction}}}{a}x + \underset{\substack{\uparrow \\ Z_y \\ \uparrow \\ \text{slope in y-direction}}}{b}y + c$$

For example, The tangent plane to $z = 2xy + 4x\sin(x+y)$ at $\left(\frac{\pi}{2}, 0, 2\pi\right)$.

$$z = 2\left(\frac{\pi}{2}\right)(0) + 4\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) = 2\pi$$

Ans: $z = \underset{\substack{\uparrow \\ Z_x}}{\bigcirc}x + \underset{\substack{\uparrow \\ Z_y}}{\bigcirc}y + \underset{\substack{\uparrow \\ \text{whatever is needed so that it goes through } \left(\frac{\pi}{2}, 0, 2\pi\right)}}{C}$

$$Z_x = 2y + 4\sin(x+y) + 4x(\cos(x+y)\cdot 1)$$
$$= 0 + 4\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} + 4\frac{\pi}{2}\left(\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0}\right) = 4.$$

$$z_y = 2\underset{\frac{\pi}{2}}{x} + 4\underset{\frac{\pi}{2}}{x}\cos(\underset{\frac{\pi}{2}}{x}+y) = \pi$$

$$\Rightarrow \quad z = 4x + \pi y + C$$

Plug in $(\frac{\pi}{2}, 0, 2\pi)$

$$2\pi = 4\left(\frac{\pi}{2}\right) + \pi(0) + C \Rightarrow C = 0$$

$$\boxed{\text{Tangent plane is } z = 4x + \pi y}$$

Several - Variables chain rule:

composition of functions.

$$\underset{(x,y,z)}{\mathbb{R}^3} \xrightarrow[f]{} \underset{(u,v)}{\mathbb{R}^2} \xrightarrow[g]{} \mathbb{R}^1$$

$$g \circ f : \underset{(x,y,z)}{\mathbb{R}^3} \longrightarrow \mathbb{R}^1$$

$$\frac{\partial(g \circ f)}{\partial x} = \underbrace{\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}}_{\text{1-variable chain rule.}} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial(g \circ f)}{\partial y} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y}$$

etc.

Example: $f(x,y,z) = (\underbrace{x^2y}_{u}, \underbrace{z(x+2y^2)}_{v})$

$\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2$ $\quad g(u,v) = 3u + 4v^2$

$\mathbb{R}^2 \xrightarrow{g} \mathbb{R}$ $\quad (g \circ f)(x,y,z) = \ldots$

$$(g\circ f)_x = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$$

$$= 3\cdot(2xy) + (8\underset{\uparrow}{v})(z)$$

$v = z(x+2y^2)$

$$= 6xy + 8z(x+2y^2)z$$

$$= \boxed{6xy + 8z^2(x+2y^2)}$$

Check: same answer if you plug u, v in from the beginning.